Published by Institute of Physics Publishing for SISSA

Received: *January 4, 2006*
Accepted: *January 17, 2006*
Published: *January 31, 2006*

NS5-branes on an ellipsis and novel marginal deformations with parafermions*

P. Marios Petropoulos[a] **and Konstadinos Sfetsos**[ba]

[a] *Centre de Physique Théorique, Ecole Polytechnique*[†]
91128 Palaiseau Cedex, France
[b] *Department of Engineering Sciences, University of Patras*
26110 Patras, Greece
E-mail: marios@cpht.polytechnique.fr, sfetsos@des.upatras.gr

Abstract: We consider NS5-branes distributed along the circumference of an ellipsis and explicitly construct the corresponding gravitational background. This provides a continuous line of deformations between the limiting cases, considered before, in which the ellipsis degenerates into a circle or into a bar. We show that a slight deformation of the background corresponding to a circle distribution into an ellipsoidal one is described by a novel non-factorizable marginal perturbation of bilinears of dressed parafermions. The latter are naturally defined for the circle case since, as it was shown in the past, the background corresponds to an orbifold of the exact conformal field theory coset model $\mathrm{SU}(2)/\mathrm{U}(1)\times\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$. We explore the possibility to define parafermionic objects at generic points of the ellipsoidal families of backgrounds away from the circle point. We also discuss a new limiting case in which the ellipsis degenerates into two infinitely stretched parallel bars and show that the background is related to the Eguchi-Hanson metric, via T-duality.

Keywords: Conformal Field Models in String Theory, p-branes.

*Research partially supported by the EU under the contracts MEXT-CT-2003-509661, MRTN-CT-2004-005104 and MRTN-CT-2004-503369.

†Unité mixte du CNRS et de l'Ecole Polytechnique, UMR 7644.

Contents

1. Introduction

The investigation of backgrounds created by brane configurations has allowed progress in various directions of string theory. In most situations, however, this progress is limited by the supergravity approximation. Even in the absence of Ramond-Ramond fluxes, exact string solutions with a clear brane interpretation are not numerous [1]–[6]. An interesting and unexpected realization of exact string background was provided in [5], by distributing N NS5-branes on a circle, either on a discrete set of points or continuously. In the latter case the geometry, antisymmetric tensor and dilaton backgrounds turn out to be T-dual to those of the $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1) \times \mathrm{SU}(2)/\,\mathrm{U}(1)$ conformal sigma model.

An exact worldsheet conformal sigma model is in general the seed for generating a wider continuous or discrete moduli space of exact string solutions. This includes, in particular, continuous deformations induced by marginal worldsheet operators.[1] Examples of this kind and in the present framework were carried out exhaustively [7]–[17], as e.g. the $\mathrm{SU}(2)/\,\mathrm{U}(1)\times\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ model, which is known to be connected to the $\mathrm{SU}(2)\times\mathrm{SL}(2,\mathbb{R})$

[1]The issue of marginal deformations in conformal models has been discussed prolifically in the literature. We cannot be exhaustive here.

JHEP01(2006)167

by an exact marginal deformation triggered by bilinears in the $\mathrm{SU}(2)$ and $\mathrm{SL}(2,\mathbb{R})$ currents. It turns out though, that along the lines of deformation one usually looses track of the brane-sources, except in a few examples such as null deformations of $\mathrm{SL}(2,\mathbb{R})$ or related models. We know for example that a T-dual version of $\mathrm{SU}(2)/\mathrm{U}(1) \times \mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ is also related to $\mathrm{SU}(2) \times \mathrm{SL}(2,\mathbb{R})$ by a marginal deformation generated by a bilinear of null combinations of $\mathrm{SU}(2)$ and $\mathrm{SL}(2,\mathbb{R})$ currents. It connects the background of a continuous distribution of NS5-branes on a circle and that of NS5-branes at a point with F1 spread in their world-volume [9, 17].

The original purpose of the present work is to present an example where an exact background is deformed by continuously deforming the distribution of the brane-sources, allowing thereby to be all the way in contact with the physical interpretation of the background fields. To be specific, we will consider NS5-branes distributed over a circle which is deformed into an ellipsis. Such a deformation is fully controllable at the supergravity level and embraces known geometries such as the Eguchi-Hanson solution [18]. The remarkable outcome of our analysis is that the departure from the circle is also driven by a marginal operator of the underlying $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1) \times \mathrm{SU}(2)/\mathrm{U}(1)$ conformal field theory. Such a result is unexpected because of the absence of currents in these coset models. Despite that, the compact parafermions of $\mathrm{SU}(2)/\mathrm{U}(1)$ can be appropriately dressed by the non-compact fields of $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ and deliver a novel kind of operator with anomalous dimension two, which is *not factorizable* in terms of holomorphic and anti-holomorphic currents. Per se this is a noticeable observation because no example of this type seems to appear elsewhere.

The supergravity solution we are describing exists away from the vicinity of the circular distribution, for any finite value of the eccentricity parameter of the ellipsis. This suggests the marginal operator be exact and always present in the worldsheet theory of the "elliptic" background.

We suggest *generalized compact and non-compact parafermions*, extensions of the ordinary $\mathrm{SU}(2)$ and $\mathrm{SL}(2,\mathbb{R})$ ones, and we show that the generic deformation is driven by dressed bilinears of these compact and non-compact generalized parafermions. The existence of this operator at any point of the elliptic deformation indicates that it is *exactly marginal* and that the corresponding worldsheet theory is exactly conformal. A rigorous proof of this statement requires the computation of the exact anomalous dimension of this *non-factorizable* operator, which is left for future work.

2. Supergravity solution for NS5-branes on an ellipsis

2.1 A constructive approach

The general form of the metric representing the gravitational field of a large number N of NS5-brane gravitational solution is

$$\mathrm{d}s^2 = \mathrm{d}s^2\left(E^{(1,5)}\right) + H \mathrm{d}s^2\left(\mathbb{R}^4\right) \ . \tag{2.1}$$

This is promoted to a solution of the field equations when it is supplemented by a closed three-form given by

$$H_{ijk} = \epsilon_{ijk}{}^{l} \partial_l H \ , \tag{2.2}$$

where the indices are lowered and raised with the flat metric in $\mathbb{R}^4$, and a dilaton field

$$e^{2\Phi} = H \ . \tag{2.3}$$

Satisfying the field equations and preserving half of maximum supersymmetry requires that the function H is harmonic in $\mathbb{R}^4$ and in general be given by

$$H(\mathbf{x}) = N \int_{\mathbb{R}^4} d^4x' \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^2} \ , \tag{2.4}$$

for some everywhere positive density function $\rho(\mathbf{x})$ normalized to 1. Since we are solely interested in the field theory-near horizon limit we have omitted the unity from the above expression for the harmonic function. If all centers coincide then the harmonic function above is simply $H = N/r^2$, where r is the radial variable in $\mathbb{R}^4$. In this case the non-trivial four-dimensional part of the background is described by the exact conformal field theory (CFT) given as the direct product of the $\mathrm{SU}(2)_N$ Wess-Zumino-Witten (WZW) model with a free boson with a background charge [1]. In the generic case in which the centers are randomly distributed the symmetry of $\mathbb{R}^4$ is completely broken and finding a CFT associated to the background seems hopeless. In [5] the background corresponding to NS5-branes uniformly distributed on a canonical N-polygon, was explicitly constructed. In this case a $\mathrm{U}(1) \times Z_N$ symmetry is preserved where the $\mathrm{U}(1)$ factor corresponds to rotations in the two-dimensional space transverse to the plane where the NS5-branes are distributed. On the other hand the Z_N factor is due to the discreteness of the distribution. In the continuum limit this factor becomes another $\mathrm{U}(1)$ and the NS5-branes are then continuously and uniformly distributed on the circumference of a circle. Moreover, using a T-duality, it was shown, that there is an exact CFT corresponding to it given by an orbifold of the $\mathrm{SU}(2)/\mathrm{U}(1) \times \mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ coset model [5] (see also [15] and for further details [16, 17]).

In this paper we will consider NS5-branes distributed on the circumference of an ellipsis with axes ℓ_1 and ℓ_2 and density distribution

$$\rho(\mathbf{x}) = \frac{1}{\pi \ell_1 \ell_2}\, \delta\left(1 - \frac{x_3^2}{\ell_1^2} - \frac{x_4^2}{\ell_2^2}\right) \delta(x_1)\delta(x_2) \ , \tag{2.5}$$

where we confine our discussion to the continuum limit. The symmetry of the background is inherited by that of the brane distribution. Therefore we expect that the background will have a reduced $\mathrm{U}(1) \times Z_2$ symmetry, where the first factor is the same as in the case of the circle distribution, namely it will correspond to rotations of the transverse to the NS5-branes plane. The Z_2 factor essentially interchanges the two axes of the elllipsis. We will be able to describe the deformation of a circle into an ellipsis as a novel non-factorizable perturbation by compact parafermion bilinears, appropriately dressed so that the perturbation is marginal. Moreover, their group-theoretical transformation properties under the broken $\mathrm{U}(1)$ symmetry factor will explain, as we shall see, the reduced symmetry from a CFT view point.

To proceed, we follow [19, 20] and parameterize in a quite general way the coordinates x_i it terms of four units vectors $\hat{x}_i$'s restricted on the unit S^3, i.e. $\sum_{i=1}^4 \hat{x}_i^2 = 1$, as

$$\mathbf{x} = \left(\sqrt{r^2-b_1}\ \hat{x}_1, \sqrt{r^2-b_2}\ \hat{x}_2, \sqrt{r^2-b_3}\ \hat{x}_3, \sqrt{r^2-b_4}\ \hat{x}_4\right) \ . \tag{2.6}$$

The constants b_i can be taken without loss of generality to be such that $b_1 \geq b_2 \geq b_3 \geq b_4$. This parameterization dictates that the radial variable $r^2 \geq b_1$. The line element on $\mathbb{R}^4$ is then written as

$$\mathrm{d}s^2\left(\mathbb{R}^4\right) = \sum_{i=1}^{4} \frac{\hat{x}_i^2}{r^2 - b_i} r^2 \mathrm{d}r^2 + \sum_{i=1}^{4} \left(r^2 - b_i\right) \mathrm{d}\hat{x}_i^2 \ , \tag{2.7}$$

where

$$f = \prod_{i=1}^{4} \left(r^2 - b_i\right) \ . \tag{2.8}$$

Also the corresponding harmonic function in $\mathbb{R}^4$ is

$$H^{-1} = N^{-1} f^{1/2} \sum_{i=1}^{4} \frac{\hat{x}_i^2}{r^2 - b_i} \ . \tag{2.9}$$

Restricting the discussion to our case, the form of the distribution density (2.5) dictates that

$$b_1 = b_2 = 0 \ , \qquad b_3 = -\ell_1^2 \ , \qquad b_4 = -\ell_2^2 \ . \tag{2.10}$$

We found convenient to parameterize the components of the unit four-vector according to the decomposition of the $\mathbf{4}$ of SO(4) under doublets and singlets of SO(2) $\times$ SO(2), i.e. $\mathbf{4} \to (\mathbf{2}, \mathbf{1}) + (\mathbf{1}, \mathbf{2})$, as

$$\begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} = \cos\theta \begin{pmatrix} \cos\tau \\ \sin\tau \end{pmatrix} \ , \qquad \begin{pmatrix} \hat{x}_3 \\ \hat{x}_4 \end{pmatrix} = \sin\theta \begin{pmatrix} \cos\psi \\ \sin\psi \end{pmatrix} \ , \tag{2.11}$$

where the ranges of the angular variables are

$$0 \leq \theta \leq \pi \ , \qquad 0 \leq \psi, \ \tau \leq 2\pi \ . \tag{2.12}$$

Also for convenience we define the functions

$$\begin{aligned} \Delta_i &= 1 + \frac{\ell_i^2}{r^2} \ , \qquad i = 1, 2 \ , \\ \Delta &= \Delta_1 \Delta_2 \cos^2\theta + \sin^2\theta \left(\Delta_1 \sin^2\psi + \Delta_2 \cos^2\psi\right) \ . \end{aligned} \tag{2.13}$$

After some tedious but nevertheless straightforward algebra we find, from eqs. (2.7) and (2.9), that the metric in $\mathbb{R}^4$ is

$$\begin{aligned} \mathrm{d}s^2\left(\mathbb{R}^4\right) = &\ \frac{\Delta}{\Delta_1 \Delta_2} \mathrm{d}r^2 + \\ &+ r^2 \Big[\left[\sin^2\theta + \cos^2\theta \left(\Delta_1 \cos^2\psi + \Delta_2 \sin^2\psi\right)\right] \mathrm{d}\theta^2 + \cos^2\theta \ \mathrm{d}\tau^2 \\ &\quad + \sin^2\theta \left(\Delta_1 \sin^2\psi + \Delta_2 \cos^2\psi\right) \mathrm{d}\psi^2 + \\ &\quad + 2 \sin\theta \cos\theta \sin\psi \cos\psi (\Delta_2 - \Delta_1) \ \mathrm{d}\theta \mathrm{d}\psi \Big] \ , \end{aligned} \tag{2.14}$$

whereas for the harmonic function we obtain

$$H^{-1} = N^{-1} \frac{\Delta r^2}{\Delta_1^{1/2} \Delta_2^{1/2}} \ . \tag{2.15}$$

JHEP01(2006)167

These suffice to compute the ten-dimensional metric from (2.1). The non-zero components of the three-form are found, from (2.2), to be

$$\begin{aligned}
H_{r\psi\tau} &= -2N\frac{\cos^2\theta\sin^2\theta}{\Delta^2 r}\left[\Delta_1^2\sin^2\psi+\Delta_2^2\cos^2\psi-\Delta_1\Delta_2\left(\Delta_1\sin^2\psi+\Delta_2\cos^2\psi\right)\right] \ ,\\
H_{r\theta\tau} &= 2N\frac{\Delta_1-\Delta_2}{\Delta^2 r}\sin\theta\cos\theta\sin\psi\cos\psi\left[\sin^2\theta+(\Delta_1+\Delta_2-\Delta_1\Delta_2)\cos^2\theta\right] \ ,\\
H_{\theta\psi\tau} &= N\frac{\cos\theta\sin\theta}{\Delta^2}\Big(\Delta_1\Delta_2(\Delta_1+\Delta_2)\cos^2\theta\\
&\qquad +\sin^2\theta\left[2\Delta_1\Delta_2+(2\Delta_1\Delta_2-\Delta_1-\Delta_2)\left(\Delta_1\sin^2\psi+\Delta_2\cos^2\psi\right)\right]\Big).
\end{aligned} \tag{2.16}$$

It will be convenient for later use to construct an antisymmetric tensor that reproduces this as its field strength. This can be computed and has non-vanishing components[2]

$$\begin{aligned}
B_{\tau\psi} &= N\frac{\Delta_1\Delta_2\cos^2\theta}{\Delta} \ ,\\
B_{\tau\theta} &= N\frac{\Delta_1-\Delta_2}{\Delta}\sin\theta\cos\theta\sin\psi\cos\psi \ .
\end{aligned} \tag{2.17}$$

As advertised above, this background has, for generic values of these parameters, a $\mathrm{U}(1)\times Z_2$ symmetry, where the former is an isometry generated by the Killing vector ∂_τ. The discrete factor is generated by the transformation $\ell_1\leftrightarrow\ell_2$, $\psi\to\pi/2-\psi$ and $\theta\to\pi-\theta$. This involves an interchange of the two axes of the ellipsis.

As in [5], new insight can be obtained by performing a T-duality transformation with respect to the Killing vector ∂_τ. Since we are dealing with NS-NS fields only, the standard rules of [21] suffice. For the metric we find

$$\begin{aligned}
\mathrm{d}s^2 &= N\Bigg\{\frac{\mathrm{d}r^2}{r^2\sqrt{\Delta_1\Delta_2}}+(\Delta_1\Delta_2)^{-1/2}\Big[\left(\Delta_1\cos^2\psi+\Delta_2\sin^2\psi\right)\ \mathrm{d}\theta^2+\frac{\Delta}{\cos^2\theta}\ \mathrm{d}\varphi^2\\
&\qquad +\Delta_1\Delta_2(\mathrm{d}\psi^2+2\mathrm{d}\varphi\mathrm{d}\psi)+2(\Delta_1-\Delta_2)\tan\theta\sin\psi\cos\psi\ \mathrm{d}\varphi\mathrm{d}\theta\Big]\Bigg\}\\
&= N\Bigg\{\frac{\mathrm{d}r^2}{r^2\sqrt{\Delta_1\Delta_2}}+(\Delta_1\Delta_2)^{-1/2}\Big[\left(\Delta_1\cos^2(\omega-\varphi)+\Delta_2\sin^2(\omega-\varphi)\right)\ \mathrm{d}\theta^2\\
&\qquad +\left(\frac{\Delta}{\cos^2\theta}-\Delta_1\Delta_2\right)\ \mathrm{d}\varphi^2+\Delta_1\Delta_2\ \mathrm{d}\omega^2\\
&\qquad +2(\Delta_1-\Delta_2)\tan\theta\sin(\omega-\varphi)\cos(\omega-\varphi)\ \mathrm{d}\varphi\mathrm{d}\theta\Bigg\} \ ,
\end{aligned} \tag{2.18}$$

where

$$\varphi=\frac{\tau}{N}\ , \qquad \omega=\psi+\frac{\tau}{N}\ . \tag{2.19}$$

The antisymmetric tensor turns out to be zero and the dilaton reads

$$\mathrm{e}^{-2\Phi}=r^2\cos^2\theta\ . \tag{2.20}$$

[2]We thank D. Zoakos for providing them.

Notice that the dilaton does not depend on the ellipsis parameters ℓ_1 and ℓ_2. Most importantly note that due to the ranges (2.12) and the redefinitions (2.19), we have the identifications

$$\omega \equiv \omega + \frac{2\pi}{N} , \qquad \varphi \equiv \varphi + \frac{2\pi}{N} . \tag{2.21}$$

Therefore the angular coordinates ω and φ have orbifold-type identifications.

2.2 Limiting cases

There are several limiting cases that are of particular interest. First, those in which the ellipsis degenerates into a circle and a bar, constructed in [5] and [22], respectively and which we review below for completeness. In addition, we will examine a new limit in which one of the axes of the ellipsis becomes very large compared to the other. In that case the ellipsis can be viewed as two infinitely long bars kept at a fixed distance apart. In these limiting cases the isometry group gets enlarged.

2.2.1 Branes on the circle

When $\ell_1^2 = \ell_2^2$ we get the solution for NS5-branes uniformly distributed on a ring, constructed in [5].[3] In this limit, using eqs. (2.14) and (2.17) we find that the background becomes

$$\begin{aligned}
\mathrm{d}s^{(0)2} &= N\left[\mathrm{d}\rho^2 + \mathrm{d}\theta^2 + \frac{1}{\Sigma}(\tanh^2\rho\mathrm{d}\tau^2 + \tan^2\theta\mathrm{d}\psi^2)\right] , \\
B^{(0)}_{\tau\psi} &= \frac{N}{\Sigma} , \\
\mathrm{e}^{-2\Phi^{(0)}} &= \Sigma\cosh^2\rho\cos^2\theta ,
\end{aligned} \tag{2.22}$$

where we changed the radial variable as $r = \ell_1 \sinh\rho$ and

$$\Sigma = \tanh^2\rho\tan^2\theta + 1 . \tag{2.23}$$

Note that for later convenience we have included an extra index in the fields to emphasize that they represent the leading order results in the expansion of the general ellipsis background around the circle case with $\ell_1^2 = \ell_2^2$. The isometry group in eq. (2.22) is $\mathrm{U}(1)\times\mathrm{U}(1)$ with Killing vectors ∂_τ and ∂_ψ, where the two factors correspond to rotations in the planes perpendicular as well well on the plane of the NS5-brane distribution. The second U(1) is a continuous approximation to the Z_N discrete symmetry that actually leaves the exact background invariant. The exact form of the background, without passing to the continuous approximation we employ here, was computed in [5]. Finally, from eq. (2.18), its T-dual is

$$\begin{aligned}
\mathrm{d}s^{(0)2} &= N\left[\mathrm{d}\rho^2 + \coth^2\rho\mathrm{d}\omega^2 + \mathrm{d}\theta^2 + \tan^2\theta\mathrm{d}\varphi^2\right] , \\
\mathrm{e}^{-2\Phi^{(0)}} &= \sinh^2\rho\cos^2\theta ,
\end{aligned} \tag{2.24}$$

with the identifications (2.21). This is the background corresponding to a Z_N orbifold of the exact CFT $\mathrm{SU}(2)/\mathrm{U}(1)\times\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$.

[3] Actually this background was constructed before in [24] in studies of the interplay between T-duality and supersymmetry, but the NS5-brane interpretation was given in [5].

JHEP01(2006)167

2.2.2 Branes on a bar

When $\ell_1 \to 0$, the NS5-branes are distributed on a bar of length $2\ell_2$ along the x_4-axis and centered in the origin. Their distribution will not be uniform in the bar. Instead, taking in (2.5) the $\ell_1 \to 0$ limit we find

$$\rho(\mathbf{x}) = \frac{1}{\pi\ell}\left(1 - \frac{x_4^2}{\ell^2}\right)^{-1/2} \Theta\left(1 - \frac{x_4^2}{\ell^2}\right) \delta(x_1)\delta(x_2)\delta(x_3) \ , \tag{2.25}$$

where we have, for notational convenience, dropped the index from ℓ_2. In this case the symmetry of the background becomes SO(3) corresponding to rotations in the three-dimensional space perpendicular to the bar. However, this symmetry does not become manifest if we just write down the background or its dual after setting $\ell_1 = 0$. The reason is that we have chosen in (2.11) to parameterize the four-unit vector in a way that a possible $\mathrm{U}(1) \times \mathrm{U}(1)$ symmetry in the background will be manifest, i.e. according to the decomposition of the $\mathbf{4}$ of SO(4) under the $\mathrm{SO}(2) \times \mathrm{SO}(2)$ subgroup as $\mathbf{4} \to (\mathbf{2}, \mathbf{1}) + (\mathbf{1}, \mathbf{2})$. In the case of a distribution on a bar the convenient parameterization is that corresponding to the decomposition of the $\mathbf{4}$ of SO(4) under the SO(3) subgroup as $\mathbf{4} \to \mathbf{3} + \mathbf{1}$. The result is given in [22] and is presented here for completeness. For this, it is convenient to use the following basis for the unit vectors $\hat{x}_i$ that define the three-sphere:

$$\begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} = \cos\theta \sin\omega \begin{pmatrix} \cos\chi \\ \sin\chi \end{pmatrix} \ , \quad \hat{x}_3 = \cos\theta\cos\omega \ , \quad \hat{x}_4 = \sin\theta \ . \tag{2.26}$$

Also for the constants b_i we choose

$$b_1 = b_2 = b_3 = 0 \ , \qquad b_4 = -\ell^2 \ . \tag{2.27}$$

The expressions following from (2.2), (2.3), (2.7) and (2.9) for the four-dimensional transverse part of the metric, the antisymmetric tensor and the dilaton fields are given explicitly by

$$\begin{aligned}
\mathrm{d}s^2 &= N\left(1 + \frac{\ell^2}{r^2}\right)^{1/2}\left[\frac{\mathrm{d}r^2}{r^2+\ell^2} + \mathrm{d}\theta^2 + \frac{r^2\cos^2\theta}{r^2+\ell^2\cos^2\theta}\left(\mathrm{d}\omega^2 + \sin^2\omega\,\mathrm{d}\chi^2\right)\right] \ , \\
B_{\omega\chi} &= N\sin\omega\left(\theta + \frac{r^2\cos\theta\sin\theta}{r^2+\ell^2\cos^2\theta}\right) \ , \\
\mathrm{e}^{2\Phi} &= \frac{\left(1+\frac{\ell^2}{r^2}\right)^{1/2}}{r^2+\ell^2\cos^2\theta} \ .
\end{aligned} \tag{2.28}$$

2.2.3 The Eguchi-Hanson metric

When one of the axis of the ellipsis becomes very large compared to the other, say $\ell_2 \gg \ell_1$, then the distribution density (2.5) becomes

$$\rho(\mathbf{x}) = \lim_{L\to\infty} \frac{1}{4L}\Theta(L - |x_4|)\left[\delta(x_3 - \ell) + \delta(x_3 + \ell)\right]\delta(x_1)\delta(x_2) \ , \tag{2.29}$$

where we have dropped for convenience the index in ℓ_1. This distribution corresponds to two parallel, infinitely extended bars each one carrying half of the branes. The limiting

JHEP01(2006)167

procedure can be directly performed in the background as it is written in (2.14)–(2.17). Indeed, let $\psi = z/\ell_2$ and then take the $\ell_2 \to \infty$ limit which corresponds to a contraction. In this limit the harmonic function in (2.15) becomes

$$H = \frac{N}{\ell_2} \frac{\sqrt{r^2 + \ell^2}}{r^2 + \ell^2 \cos^2\theta} \ . \tag{2.30}$$

In order to make sense of the limit we have to absorb into the overall scale the factor $1/\ell_2$. Using the parameterization

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = r\cos\theta \begin{pmatrix} \cos\tau \\ \sin\tau \end{pmatrix} \ , \qquad x_3 = \sqrt{r^2+\ell^2}\sin\theta \ , \qquad x_4 = z\sin\theta \ , \tag{2.31}$$

we find the useful relations

$$\begin{aligned} &R_+ + R_- = 2\sqrt{r^2+\ell^2} \ , \qquad R_+ - R_- = 2\ell\sin\theta \ , \qquad R_+R_- = r^2 + \ell^2\cos^2\theta \ , \\ &R_\pm = \sqrt{x_1^2 + x_2^2 + (x_3 \pm \ell)^2} \ . \end{aligned} \tag{2.32}$$

The result is a metric of the form

$$\mathrm{d}s^2 = H\left(\mathrm{d}x_4^2 + \mathrm{d}\mathbf{x}^2\right) \ , \qquad H = \frac{N}{2}\left(\frac{1}{R_+} + \frac{1}{R_-}\right) \ , \tag{2.33}$$

where we have defined the three-dimensional vector $\mathbf{x} = (x_1, x_2, x_3)$ The antisymmetric tensor two-form reads:

$$\begin{aligned} B &= \mathrm{d}x_4 \wedge \omega \cdot \mathrm{d}\mathbf{x} \ , \\ \omega &= \frac{N/2}{x_1^2 + x_2^2}\left(\frac{x_3+\ell}{R_+} + \frac{x_3-\ell}{R_-}\right) (-x_2, x_1, 0) \ , \end{aligned} \tag{2.34}$$

where we have also absorbed a factor of $1/\ell_2$ into the overall scale as in the case of the metric above. Its T-dual with respect to the Killing vector $\partial/\partial x_4$ is nothing but the purely gravitational Eguchi-Hanson metric [18] written in its Gibbons-Hawking [25] two-center form

$$\mathrm{d}s^2 = H^{-1}(\mathrm{d}x_4 + \omega\cdot\mathrm{d}\mathbf{x})^2 + H\mathrm{d}\mathbf{x}^2 \ . \tag{2.35}$$

Regularity of the metric at the two *nuts* at $(\mathbf{x}, x_4) = (\mathbf{0}, \pm\ell)$ requires the identification $x_4 \equiv x_4 + 4\pi N$. The symmetry of the Eguchi-Hanson metric [18] is $\mathrm{SU}(2) \times \mathrm{U}(1)/Z_2$. This is not manifest in the two-center form of the metric but there are global-symmetry considerations for that as well as an explicit transformation [23] that transforms the metric into the form presented initially by Eguchi-Hanson. The background (2.33)–(2.34) inherits the same symmetry due to the fact that $\partial/\partial x_4$ is a translational Killing vector. We note also that (2.35) is not the background corresponding to the metric (2.18) and the dilaton (2.20) after the contraction limit is taken. The latter gives a background that is not purely gravitational since there is still a non-trivial dilaton, as it can be easily seen. It can be obtained from the Eguchi-Hanson metric if the T-duality is performed with respect to a second U(1) corresponding to rotations in the (x_1, x_2) plane.

JHEP01(2006)167

3. The deformation from the circle to the ellipsis

In this section we will consider ellipses that deviate slightly from circles and similarly for their T-duals. We will be interested in characterizing the perturbation in terms of the corresponding CFT. We will explicitly show that the perturbation can be written as bilinears in parafermions, which are the natural chiral and anti-chiral objects in the SU(2)/ U(1) coset model, dressed appropriately so that the perturbation is marginal.

3.1 Expansion around the circle

First we expand the general ellipsis background around the circle limiting case when $\ell_1^2 = \ell_2^2$, eq. (2.22).[4] From eqs. (2.14) and (2.15) we compute the leading order correction to the background (2.22) to be for the metric

$$\mathrm{d}s^{(1)2} = \frac{\ell_1^2 - \ell_2^2}{2\ell_1^2 \cosh^2 \rho} \Bigg[\cos 2\psi \left(\mathrm{d}\theta^2 + \tan^2\theta \frac{\tanh^4 \rho \mathrm{d}\tau^2 - \mathrm{d}\psi^2}{\Sigma^2} \right) - 2\sin 2\psi \frac{\tan\theta}{\Sigma} \mathrm{d}\psi \mathrm{d}\theta + \mathrm{d}\rho^2 + \frac{\tanh^2 \rho}{\Sigma^2} \left(\mathrm{d}\tau^2 - \tan^2\theta \, \mathrm{d}\psi^2 \right) \Bigg], \quad (3.1)$$

where, as before, we have changed to a new radial variable as $\rho = \ell_1 \sinh \rho$. For the antisymmetric tensor, the leading correction is

$$\begin{aligned} B^{(1)}_{\tau\psi} &= -\frac{\ell_1^2 - \ell_2^2}{2\ell_1^2 \cosh^2 \rho} (1 - \cos 2\psi) \frac{\tanh^2 \rho \tan^2 \theta}{\Sigma^2} , \\ B^{(1)}_{\tau\theta} &= \frac{\ell_1^2 - \ell_2^2}{2\ell_1^2 \cosh^2 \rho} \sin 2\psi \frac{\tanh^2 \rho \tan\theta}{\Sigma} . \end{aligned} \quad (3.2)$$

Similarly, for the T-dual background the leading order correction to the metric is

$$\mathrm{d}s^{(1)2} = \frac{\ell_1^2 - \ell_2^2}{2\ell_1^2 \cosh^2 \rho} \Bigg[\mathrm{d}\rho^2 - \coth^2 \rho \mathrm{d}\omega^2 + \cos 2(\omega - \varphi) \left(\mathrm{d}\theta^2 - \tan^2\theta \mathrm{d}\varphi^2 \right) + 2\sin 2(\omega - \varphi) \tan\theta \mathrm{d}\varphi \mathrm{d}\theta \Bigg]. \quad (3.3)$$

We observe that the corrections have terms that are invariant under shifts of ψ. These terms can be absorbed by a simple reparametrization of ρ. Indeed, let

$$\rho \to \rho - \frac{\ell_1^2 - \ell_2^2}{4} \tanh \rho \, . \quad (3.4)$$

This induces a correction to the background (2.22) which should be combined with (3.1) and (3.2). In this way we find that the correction to the metric (3.1) becomes ("tot" stands for the combined deformation and reparameterization)

$$\mathrm{d}s^{(1)2}|_{\text{tot}} = \frac{\ell_1^2 - \ell_2^2}{2\ell_1^2 \cosh^2 \rho} \times \left[\cos 2\psi \left(\mathrm{d}\theta^2 + \tan^2\theta \frac{\tanh^4 \rho \mathrm{d}\tau^2 - \mathrm{d}\psi^2}{\Sigma^2} \right) - 2\sin 2\psi \frac{\tan\theta}{\Sigma} \mathrm{d}\psi \mathrm{d}\theta \right]. \quad (3.5)$$

[4] We also omit, for notational convenience, writing explicitly the overall factor N. This will be done consistently for the rest of this paper.

Also the correction to the antisymmetric tensor (3.2) reads:

$$
\begin{aligned}
B^{(1)}_{\tau\psi}|_{\rm tot} &= \frac{\ell_1^2-\ell_2^2}{2\ell_1^2\cosh^2\rho}\cos 2\psi\frac{\tanh^2\rho\tan^2\theta}{\Sigma^2} \ , \\
B^{(1)}_{\tau\theta}|_{\rm tot} &= \frac{\ell_1^2-\ell_2^2}{2\ell_1^2\cosh^2\rho}\sin 2\psi\frac{\tanh^2\rho\tan\theta}{\Sigma} \ .
\end{aligned}
\tag{3.6}
$$

For the T-dual background expanding around the $\ell_1^2=\ell_2^2$ case (2.18) we obtain

$$
{\rm d}s^{(1)2}|_{\rm tot} = \frac{\ell_1^2-\ell_2^2}{2\ell_1^2\cosh^2\rho}\left[\cos 2(\omega-\varphi)\left({\rm d}\theta^2-\tan^2\theta{\rm d}\varphi^2\right)+2\sin 2(\omega-\varphi)\tan\theta{\rm d}\varphi{\rm d}\theta\right]. \tag{3.7}
$$

Note also that for the variable r this implies $\delta r\sim r$, so that the dilaton is just shifted by a constant.

3.2 On parafermions and $\mathrm{SL}(2,\mathbb{R})$ primaries

Coset theories have natural chirally and anti-chirally conserved objects, which are called parafermions due to their non-trivial braiding properties. Let's denote them as

$$
\Psi^{(1)}_\pm \ , \ \bar\Psi^{(1)}_\pm \in \frac{\mathrm{SU}(2)_N}{\mathrm{U}(1)} \qquad \text{and} \qquad \Psi^{(2)}_\pm \ , \ \bar\Psi^{(2)}_\pm \in \frac{\mathrm{SL}(2,\mathbb{R})_N}{\mathrm{U}(1)} \ . \tag{3.8}
$$

The first pair are also known as compact parafermions and were first introduced in [26]. Their non-compact counterparts are denoted by the second pair and were found in [27]. They obey the chiral and anti-chiral conservation laws

$$
\partial_-\Psi^{(i)}_\pm = 0 \ , \qquad \partial_+\bar\Psi^{(i)}_\pm = 0 \ , \qquad i=1,2 \tag{3.9}
$$

and generate infinite dimensional chiral algebras. Consider next the semiclassical expressions for the chiral parafermions in terms of space variables [28]

$$
\begin{aligned}
\Psi^{(1)}_\pm &= (\partial_+\theta \mp i\tan\theta\partial_+\varphi){\rm e}^{\mp i(\varphi+\phi_1)} \ , \\
\Psi^{(2)}_\pm &= (\partial_+\rho \mp i\coth\rho\partial_+\omega){\rm e}^{\mp i(\omega+\phi_2)} \ .
\end{aligned}
\tag{3.10}
$$

The phases are

$$
\begin{aligned}
\phi_1 &= -\frac{1}{2}\int^{\sigma^+} J^1_+{\rm d}\sigma^+ + \frac{1}{2}\int^{\sigma^-} J^1_-{\rm d}\sigma^- \ , \\
\phi_2 &= -\frac{1}{2}\int^{\sigma^-} J^2_-{\rm d}\sigma^- + \frac{1}{2}\int^{\sigma^+} J^2_+{\rm d}\sigma^+ \ ,
\end{aligned}
\tag{3.11}
$$

where $J^1_\pm$ and $J^2_\pm$ are the current components associated to the Killing vectors ∂_φ and ∂_ω, respectively[5]

$$
J^1_\pm = \tan^2\theta\partial_\pm\varphi \ , \qquad J^2_\pm = \coth^2\rho\partial_\pm\omega \ . \tag{3.12}
$$

[5]Recall that in general the currents corresponding to a Killing vector ξ read

$$
J^\xi_\pm = \xi^\mu\left(G_{\mu\nu}\mp B_{\mu\nu}\right)\partial_\pm x^\nu
$$

and obey, on shell, the conservation law $\partial_+J^\xi_- + \partial_-J^\xi_+ = 0$.

JHEP01(2006)167

Note that the phases obey on-shell the condition

$$\partial_+\partial_-\phi_i = \partial_-\partial_+\phi_i \ , \qquad i = 1,2 \ , \tag{3.13}$$

and are well defined, due to the first of the classical equations of motion

$$\partial_+ J^1_- + \partial_- J^1_+ = 0 \ , \qquad \partial_+\partial_-\theta - \frac{\sin\theta}{\cos^3\theta}\partial_+\varphi\partial_-\varphi = 0 \ , \tag{3.14}$$

for the $\mathrm{SU}(2)/\,\mathrm{U}(1)$ coset and

$$\partial_+ J^2_- + \partial_- J^2_+ = 0 \ , \qquad \partial_+\partial_-\rho + \frac{\cosh\rho}{\sinh^3\rho}\partial_+\omega\partial_-\omega = 0 \ , \tag{3.15}$$

for the $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ coset. The same equations of motion ensure that for the anti-chiral ones the corresponding expressions are

$$\begin{aligned} \bar\Psi^{(1)}_\pm &= (\partial_-\theta \pm i\tan\theta\partial_-\varphi)\ \mathrm{e}^{\pm i(\varphi-\phi_1)} \ , \\ \bar\Psi^{(2)}_\pm &= (\partial_-\rho \pm i\coth\rho\partial_-\omega)\ \mathrm{e}^{\pm i(\omega-\phi_2)} \ . \end{aligned} \tag{3.16}$$

The full set of the classical equations of motion guaranties also the conservation laws (3.9). For the readers who are willing to check that, we note that using current conservation we have that $\int^{\sigma^\pm}\partial_\mp J^i_\pm d\sigma^\pm = -J^i_\mp$. Therefore, $\partial_\pm\phi_1 = \mp J^1_\pm$ and $\partial_\pm\phi_2 = \pm J^2_\pm$. Due to the non-local phases attached to them, the classical parafermions are non-local objects and have non-trivial braiding properties. The tentative reader will notice that these phases are different from the ones used in the computation of the Poisson algebra for the classical parafermions (see [28]). In this computation it was important that the parafermions were not dependent on the past history of the "time" variable ($-$ for the $\Psi^{(i)}_\pm$ and $+$ for the $\bar\Psi^{(i)}_\pm$). Hence, for $\Psi^{(i)}_\pm$ the phases were chosen to be twice the terms involving J^i_+ in (3.11). Similarly, for $\bar\Psi^{(i)}_\pm$ the phases were chosen to be twice the terms involving J^i_- in (3.11). This does not affect the expressions for the derivatives $\partial_\pm\phi_{1,2}$ given above and therefore their conservation properties (3.9) remain intact. The choice of phases we have made facilitates the computations of this paper.

The chiral and anti-chiral energy-momentum tensors for the σ-model corresponding to the background (2.24) can be written as a sum of terms involving the corresponding chiral and anti-chiral parafermions. In particular, we have that

$$T_{++} = g_{ij}\partial_+ x^i\partial_+ x^j = \Psi^{(1)}_+\Psi^{(1)}_- + \Psi^{(2)}_+\Psi^{(2)}_- \ , \tag{3.17}$$

obeying $\partial_- T_{++} = 0$, as it should be and similarly for T_{--}.

We are interested in constructing chiral and anti-chiral objects for the T-dual to (2.24) background, namely for the background (2.22). Under the T-duality transformation we have the following map of worldsheet derivatives

$$\partial_\pm\varphi \to \pm\tilde J^1_\pm \ , \tag{3.18}$$

where we have defined

$$\tilde J^1_\pm = \frac{1}{\Sigma}\left(\tanh^2\rho\partial_\pm\tau \mp \partial_\pm\psi\right) \ . \tag{3.19}$$

For convenience we also define the currents

$$\tilde{J}^2_{\pm} = \frac{1}{\Sigma}\left(\tan^2\theta\partial_{\pm}\psi \pm \partial_{\pm}\tau\right) \tag{3.20}$$

and note the transformation of the worldsheet derivatives

$$\partial_{\pm}\omega \to \tanh^2\rho\tilde{J}^2_{\pm} \ . \tag{3.21}$$

The equations of motion that follow from varying τ and ψ are

$$\partial_+\tilde{J}^i_- + \partial_-\tilde{J}^i_+ = 0 \ , \qquad i = 1,2 \ , \tag{3.22}$$

whereas those from varying θ and ρ are

$$\begin{aligned} &\partial_+\partial_-\theta + \frac{\sin\theta}{\cos^3\theta}\tilde{J}^1_+\tilde{J}^1_- = 0 \ , \\ &\partial_+\partial_-\rho + \frac{\sinh\rho}{\cosh^3\rho}\tilde{J}^2_+\tilde{J}^2_- = 0 \ . \end{aligned} \tag{3.23}$$

Using the above, we determine the transformation of the phase factors appearing in the various expressions for the parafermions, under the action of T-duality:

$$\begin{aligned} &\varphi + \phi_1 \to -\tau - \tilde{\phi} \ , \qquad \omega + \phi_2 \to \psi - \tilde{\phi} \ , \\ &\varphi - \phi_1 \to \tau + \bar{\tilde{\phi}} \ , \qquad \omega - \phi_2 \to \psi + \bar{\tilde{\phi}} \ , \end{aligned} \tag{3.24}$$

where the new phases are

$$\begin{aligned} \tilde{\phi} &= -\frac{1}{2}\int^{\sigma^+}\left(\tilde{J}^1_+ + \tilde{J}^2_+\right)\mathrm{d}\sigma^+ + \frac{1}{2}\int^{\sigma^-}\left(\tilde{J}^1_- + \tilde{J}^2_-\right)\mathrm{d}\sigma^- \ , \\ \bar{\tilde{\phi}} &= \frac{1}{2}\int^{\sigma^+}\left(\tilde{J}^1_+ - \tilde{J}^2_+\right)\mathrm{d}\sigma^+ + \frac{1}{2}\int^{\sigma^-}\left(-\tilde{J}^1_- + \tilde{J}^2_-\right)\mathrm{d}\sigma^- \ . \end{aligned} \tag{3.25}$$

In this way we find that the chiral and the anti-chiral parafermions become

$$\begin{aligned} \tilde{\Psi}^{(1)}_{\pm} &= \left(\partial_+\theta \mp i\tan\theta\tilde{J}^1_+\right)\ \mathrm{e}^{\pm i(\tau+\tilde{\phi})} \ , \\ \tilde{\Psi}^{(2)}_{\pm} &= \left(\partial_+\rho \mp i\tanh\rho\tilde{J}^2_+\right)\ \mathrm{e}^{\mp i(\psi-\tilde{\phi})} \ , \end{aligned} \tag{3.26}$$

and

$$\begin{aligned} \bar{\tilde{\Psi}}^{(1)}_{\pm} &= \left(\partial_-\theta \mp i\tan\theta\tilde{J}^1_-\right)\ \mathrm{e}^{\pm i(\tau+\bar{\tilde{\phi}})} \ , \\ \bar{\tilde{\Psi}}^{(2)}_{\pm} &= \left(\partial_-\rho \pm i\tanh\rho\tilde{J}^2_-\right)\ \mathrm{e}^{\pm i(\psi+\bar{\tilde{\phi}})} \ , \end{aligned} \tag{3.27}$$

respectively. We may check that, on-shell, these parafermions are indeed chiral and anti-chiral, provided the equations of motion (3.22) and (3.23) for the background (2.22) are obeyed.

As an additional consistency check note that the energy-momentum tensor for the σ-model corresponding to the background (2.22) can be written as a sum of terms involving the corresponding parafermions. In particular,

$$\tilde{T}_{++} = \tilde{g}_{ij}\partial_+x^i\partial_+x^j = \tilde{\Psi}^{(1)}_+\tilde{\Psi}^{(1)}_- + \tilde{\Psi}^{(2)}_+\tilde{\Psi}^{(2)}_- \ , \tag{3.28}$$

as it should be and similarly for $\tilde{T}_{--}$.

For equal levels N of the SU(2) and $SL(2,\mathbb{R})$ factors in the coset models, it is well known [26, 27] that the conformal dimensions of the parafermions are $1 \mp 1/N$, the two different signs corresponding to the compact and non-compact ones, respectively. It is convenient for this paper to understand them in terms of the energy-momentum tensor of the theory. Consider first the compact case in which the energy-momentum tensor is given by the difference of the Sugawara constructions of the two energy-momentum tensors for the group SU(2) and its U(1) subgroup. Hence for any operator $\Delta_{\mathrm{SU(2)/\,U(1)}} = \Delta_{\mathrm{SU(2)}} - \Delta_{\mathrm{U(1)}}$, where $\Delta_{\mathrm{U(1)}} = m^2/N$ with m the U(1) charge. The parafermions originate from the $J_\pm$ SU(2) currents which have $\Delta_{\mathrm{SU(2)}} = 1$. Since their charge eigenvalue is $m = \pm 1$, we have that $\Delta_{\mathrm{U(1)}} = 1/N$. Then the result mentioned above follows. The difference in the sign for the non-compact parafermions is due to the fact that in this case $\Delta_{\mathrm{U(1)}} = -m^2/N$.

As a final piece of information consider the construction of $SL(2,\mathbb{R})$ WZW theory affine primaries as composites of group elements. Let's define the group element $g \in SL(2,\mathbb{R})$ in the spinor representation as

$$g = \mathrm{e}^{\frac{i}{2}\theta_{\mathrm{L}}\sigma_2}\mathrm{e}^{\rho\sigma_1}\mathrm{e}^{\frac{i}{2}\theta_{\mathrm{R}}\sigma_2} \ . \tag{3.29}$$

The $SL(2,\mathbb{R})$ algebra generators in terms of Pauli matrices are $J_0 = \sigma_2/2$, $J_\pm = \pm(\sigma_1 \mp i\sigma_3)/2$. Using these, we define four elements g_{ab}, $a, b = \pm$ as

$$g_{\pm\pm} = \mathrm{Tr}(R_\pm g) \ , \qquad g_{\pm\mp} = \pm\, \mathrm{Tr}(J_\pm g) \ , \tag{3.30}$$

where $R_\pm = \frac{1}{2}(\mathbb{1} \pm \sigma_2)$. Explicitly we have that

$$g_{\pm\pm} = \cosh\rho \ \mathrm{e}^{\pm i(\theta_{\mathrm{L}}+\theta_{\mathrm{R}})/2} \ , \qquad g_{\pm\mp} = \sinh\rho \ \mathrm{e}^{\mp i(\theta_{\mathrm{L}}-\theta_{\mathrm{R}})/2} \ . \tag{3.31}$$

These transform in the $(\frac{1}{2},\frac{1}{2})$ representation of $SL(2,\mathbb{R})_{\mathrm{L}} \times SL(2,\mathbb{R})_{\mathrm{R}}$ with U(1) charges $(\pm\frac{1}{2},\pm\frac{1}{2})$, in all four combinations, in accordance with their index. The explicit transformation rules referring to $SL(2,\mathbb{R})_{\mathrm{L}}$ are

$$\begin{aligned}
&\delta_0 g_{\pm\pm} = \pm\frac{1}{2} g_{\pm\pm} \ , &&\delta_0 g_{\pm\mp} = \pm\frac{1}{2} g_{\pm\mp} \ ,\\
&\delta_- g_{++} = -g_{-+} \ , &&\delta_+ g_{++} = 0 \ ,\\
&\delta_- g_{+-} = -g_{--} \ , &&\delta_+ g_{+-} = 0 \ ,\\
&\delta_- g_{-+} = 0 \ , &&\delta_+ g_{-+} = g_{++} \ ,\\
&\delta_- g_{--} = 0 \ , &&\delta_+ g_{--} = \delta_{+-}
\end{aligned} \tag{3.32}$$

and similarly for transformations with respect to $SL(2,\mathbb{R})_{\mathrm{R}}$. We may construct other irreducible representations by forming composites of these elements. In particular, consider the four simple composite objects

$$A_{++} = \frac{1}{g_{--}^2} \ , \qquad A_{+-} = \frac{1}{g_{-+}^2} \ , \qquad A_{-+} = \frac{1}{g_{+-}^2} \ , \qquad A_{--} = \frac{1}{g_{++}^2} \ , \tag{3.33}$$

which will be very useful as we will soon see. They clearly have charges $(1,1)$, $(1,-1)$, $(-1,1)$ and $(-1,-1)$, respectively. Using (3.32) we see that $\delta_- A_{++} = \delta_- A_{+-} = 0$ and $\delta_+ A_{--} = \delta_+ A_{-+} = 0$. Hence, A_{++} forms a lowest weight representation for $\mathrm{SL}(2,\mathbb{R})_\mathrm{L}$ as well as for $\mathrm{SL}(2,\mathbb{R})_\mathrm{R}$. The other members of the representation are obtained by repeatedly acting with δ_+ for the left and the right $\mathrm{SL}(2,\mathbb{R})$ factor and the charges of these states are accordingly increased. In terms of $\mathrm{SL}(2,\mathbb{R})$ representation theory this state belongs to the positive discrete series D^+ with $j = \bar{j} = 0$ and U(1) charges $m = \bar{m} = 1$ for both $\mathrm{SL}(2,\mathbb{R})$ factors. Similarly, A_{+-} forms a lowest-weight representation for $\mathrm{SL}(2,\mathbb{R})_\mathrm{L}$ and a highest-weight representation for $\mathrm{SL}(2,\mathbb{R})_\mathrm{R}$. The other members of the representation are obtained by repeatedly acting with δ_+ for the left and δ_- for the right $\mathrm{SL}(2,\mathbb{R})$ factor. In terms of $\mathrm{SL}(2,\mathbb{R})$ representation theory this state belongs to the positive discrete series D^+ with $j = 0$ and $m = 1$ for $\mathrm{SL}(2,\mathbb{R})_\mathrm{L}$ and to the negative discrete series D^- with $j = 0$ and $\bar{m} = -1$ for $\mathrm{SL}(2,\mathbb{R})_\mathrm{R}$. It is convenient to use for the four states in (3.33) a notation that directly relates to their labelling according to the $\mathrm{SL}(2,\mathbb{R})$ representation theory we have mentioned. In an obvious notation we have[6]

$$\begin{aligned} \Phi^{\mathrm{lw,lw}}_{0,1,1} &= \frac{1}{g^2_{--}} \,, \qquad & \Phi^{\mathrm{lw,hw}}_{0,1,-1} &= \frac{1}{g^2_{-+}} \,, \\ \Phi^{\mathrm{hw,lw}}_{0,-1,1} &= \frac{1}{g^2_{+-}} \,, \qquad & \Phi^{\mathrm{hw,hw}}_{0,-1,-1} &= \frac{1}{g^2_{++}} \,. \end{aligned} \tag{3.34}$$

Since $j = 0$ for all of these states, their conformal dimensions, given by $-j(j+1)/(N-2)$, equal zero.

The background for the $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ follows by considering the vector gauging of a U(1) subgroup generated by σ_2, as it can be seen from (3.29). In a standard procedure, the configuration space is reduced by one dimension by a gauge fixing. Under the vector transformation $\delta\theta_\mathrm{L} = -\epsilon$ and $\delta\theta_\mathrm{R} = \epsilon$, the $g_{\pm\pm}$ parameters, as given by (3.31), are invariant whereas $g_{\pm\mp}$ are not. The parafermions are by construction gauge invariant, essentially due to the dressing of the $J_\pm$ currents with the Wilson lines attached to them. Because of the gauge invariance we choose a unitary gauge, which amounts to setting $\theta_\mathrm{L} = \theta_\mathrm{R} = \omega$. The corresponding background is given by the relevant part in (2.24). Then

$$g_{\pm\pm}|_{\mathrm{g.f.}} = \cosh\rho\ \mathrm{e}^{\pm i\omega} \,, \qquad g_{\pm\mp}|_{\mathrm{g.f.}} = \sinh\rho \tag{3.35}$$

are the gauged-fixed elements. Since they have U(1) charges ± 1, the states have conformal dimensions equal to $1/N$.

Under T-duality the factor ω in the states (3.34) transforms as $\omega \to \psi - \frac{1}{2}(\tilde{\phi} - \bar{\tilde{\phi}})$.

3.3 The deformation around the circle as a marginal perturbation

Using the above formalism, it is easy to show that the correction (3.7) can be reproduced by adding to the sigma-model action based on the unperturbed background (2.24) the term

$$\delta\mathcal{L}|_{\delta\ell_2} = \frac{\ell_1^2 - \ell_2^2}{4\,\ell_1^2} \left(\Phi^{\mathrm{lw,lw}}_{0,1,1} \Psi^{(1)}_+ \bar{\Psi}^{(1)}_- + \Phi^{\mathrm{hw,hw}}_{0,-1,-1} \Psi^{(1)}_- \bar{\Psi}^{(1)}_+ \right)$$

[6] More general representations were worked out in a similar fashion in [29] with findings that agree with ours.

$$= \frac{\ell_1^2 - \ell_2^2}{4\,\ell_1^2\cosh^2\rho}\left(\Psi_+^{(1)}\bar{\Psi}_-^{(1)}\mathrm{e}^{2i\omega} + \Psi_-^{(1)}\bar{\Psi}_+^{(1)}\mathrm{e}^{-2i\omega}\right) \tag{3.36}$$

as a perturbation. This is a $(1,1)$ marginal perturbation since the conformal dimensions add up to one $(1/N + (1-1/N) = 1)$. There is however, a delicate point to consider here. In order to preserve superconformal invariance, the central charge of the theory has to be $c = 6$, out of which two units are due to the free fermions. Hence, the bosonic part which consists of the cosets we have been working with has to provide the other four units. This, however, is only possible if the level of the $\mathrm{SU}(2)$ factor is N and that of the $\mathrm{SL}(2,\mathbb{R})$ factor $N+4$ [30]. The supergravity solution is obviously insensitive to this difference in the levels, since N is assumed to be very large for the supergravity description to be valid. The above modification affects the conformal dimension of the operators in (3.34) which becomes $1/(N+4)$ and therefore the total dimension would receive $1/N^2$ corrections, seizing the perturbation (3.36) from being marginal unless we deal with a bosonic theory. The complete resolution of this issue is postponed for future work, but we believe that it relies into properly taking into account the fermionic degrees of freedom. The perturbation (3.36) is not factorizable. This is of immense importance as we know of no other such example in the literature. The rôle of the states in (3.34) in dressing the parafermion bilinear in the perturbation (3.36) is very important. To appreciate this point we note that, if the perturbation was purely due to parafermion bilinears, it would not be marginal and obviously it would not couple the two factors in the $\mathrm{SU}(2)/\mathrm{U}(1) \times \mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ model. Instead, it was shown in [31] that the perturbed $\mathrm{SU}(2)/\mathrm{U}(1)$ conformal model is integrable and massive, and that it is related to the $O(3)$ model or it flows in the infrared to the minimal models, depending on the details of the perturbation.

In the dual background the term (3.36) may be interpreted using the tilded parafermions as

$$\begin{aligned} \delta\tilde{\mathcal{L}}|_{\delta\ell_2} &= \frac{\ell_1^2 - \ell_2^2}{4\,\ell_1^2}\left(\tilde{\Phi}^{\mathrm{lw,lw}}_{0,1,1}\tilde{\Psi}_+^{(1)}\bar{\tilde{\Psi}}_-^{(1)} + \tilde{\Phi}^{\mathrm{hw,hw}}_{0,-1,-1}\tilde{\Psi}_-^{(1)}\bar{\tilde{\Psi}}_+^{(1)}\right) \\ &= \frac{\ell_1^2 - \ell_2^2}{4\,\ell_1^2\cosh^2\rho}\left(\tilde{\Psi}_+^{(1)}\bar{\tilde{\Psi}}_-^{(1)}\mathrm{e}^{2i\psi - i(\tilde{\phi}-\bar{\tilde{\phi}})} + \tilde{\Psi}_-^{(1)}\bar{\tilde{\Psi}}_+^{(1)}\mathrm{e}^{-2i\psi + i(\tilde{\phi}-\bar{\tilde{\phi}})}\right)\,. \end{aligned} \tag{3.37}$$

These terms precisely reproduce (3.5) and (3.6). The deformations at hand of the background generated by NS5-branes distributed over a circle are of a new type, as advertised previously. In particular, the fact that under the symmetry generated by the Killing vector ∂_ψ, the parafermions transform as a doublet and the fact that (3.36) is not a singlet, explains the breaking of the $\mathrm{U}(1)$ symmetry associated with the plane where the NS5-branes are distributed which occurs as the circle is deformed into an ellipsis.

It is legitimate to ask at this point why the NS5-branes, continuously distributed over an ellipsis should give still rise to an exact theory as for the circle. The main argument in favor of this is that the background at hand solves the supergravity equations for any value of ℓ_1/ℓ_2, with the point $\ell_1 = \ell_2$ being an exact conformal field theory dual to $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1) \times \mathrm{SU}(2)/\mathrm{U}(1)$. Furthermore, departure from $\ell_1 = \ell_2$ is triggered by a conformal operator of dimension $(1,1)$. One therefore expects that an underlying exact CFT must exist beyond $\ell_1 = \ell_2$. Arguing in favor of that is the purpose of next section.

JHEP01(2006)167

4. The general perturbation of the ellipsis

4.1 Towards generalized parafermions

We have shown so far that one can move the locus where the NS5-branes are distributed, from a circle of radius ℓ to an ellipsis with axes ℓ_1, ℓ_2. The corresponding metric, antisymmetric tensor and dilaton background solve, by construction, the supergravity equations. An important question is whether this background is perturbatively exact, i.e. if it receives $\alpha' \sim 1/N$-corrections. Let's recall that backgrounds corresponding to two-dimensional σ-models with $(4,4)$ worldsheet generalized extended supersymmetry [32 − 34] do not receive α'-corrections at any order in perturbation theory. This is due to the large extended supersymmetry which makes the corresponding counterterms vanish [32, 35].

The background of the form given in eq. (2.1) has $(4,4)$ extended worldsheet supersymmetry with the explicit expressions of the complex structures given in [24]:

$$F_i^{\pm} = H\left(\frac{1}{2}\epsilon_{ijk}\ \mathrm{d}x_j \wedge \mathrm{d}x_k \ \pm\ \mathrm{d}x_i \wedge \mathrm{d}x_4\right) , \qquad i = 1,2,3 \ . \tag{4.1}$$

The absence of $1/N$ corrections does not imply the absence of non-perturbative corrections. In the case of the circle background its exact form, before the continuum limit for the distribution was taken, contains explicitly such corrections which, however, are washed out in the continuum limit [5]. These corrections should be discussed in the spirit of similar studies for the discretized version of NS5-branes distributed uniformly on an infinite straight line in [36]. Such non-perturbative corrections are expected to arise for the background corresponding to the ellipsoidal distribution. Even constructing the harmonic function beyond the continuum limit (2.15) is however challenging in this case.

Starting from the circle and going to an almost circular ellipsis with axes ℓ_1 and $\ell_2 = \ell_1 + \delta\ell$ is possible by means of $(1,1)$ marginal operators based on the compact parafermions of $\mathrm{SU}(2)/\mathrm{U}(1)$ appropriately dressed with operators of the non-compact side $(\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1))$. Those are available in the exact conformal field theory of the circle and we have exhibited them in section 3.3.

A natural question is whether or not we can integrate the above marginal deformation for finite values of the difference $\ell_2 - \ell_1$. The usual argument for verifying the integrability of marginal operators [37] is valid in the case of left-right factorized ones, namely bilinears in holomorphic and anti-holomorphic currents. This argument does not apply, however, in the $(1,1)$ operators appearing in eqs. (3.36) or (3.37) which *are not factorized* because of their non-compact dressing (3.34). We note that there are many examples of integrated marginal perturbation when the latter is a current-current perturbation. This was done explicitly for the background corresponding to the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ WZW in [38].

Finding the general conditions under which a non-factorized $(1,1)$ operator is integrable is a task that goes beyond the scope of the present work. We will adopt another strategy and search directly in the background of the ellipsis how to recast a perturbation $\ell_1 \to \ell_1 + \delta\ell_1$, $\ell_2 \to \ell_2 + \delta\ell_2$ in the manner (3.36) or (3.37). As for the circle, the operators driving such a perturbation are not expected to be factorized and indeed they are not. We will see, however, that they turn out to exhibit an interesting *generalized parafermionic structure* valid at any ℓ_1 and ℓ_2.

We will concentrate on the background (2.18) were the antisymmetric tensor vanishes and the dilaton does not depend on the parameters (ℓ_1, ℓ_2), instead of its T-dual version (2.14)–(2.17). Let us rewrite the metric in a suggestive form

$$\begin{aligned} \mathrm{d}s^2 = & \left(\frac{\mathrm{d}r}{r\,(\Delta_1\Delta_2)^{1/4}} + i\,(\Delta_1\Delta_2)^{1/4}\,\mathrm{d}\omega \right) \left(\frac{\mathrm{d}r}{r\,(\Delta_1\Delta_2)^{1/4}} - i\,(\Delta_1\Delta_2)^{1/4}\,\mathrm{d}\omega \right) \\ & + \left(\left(\frac{\Delta_1}{\Delta_2}\right)^{1/4} [\cos\psi\,\mathrm{d}\theta + \tan\theta\sin\psi\,\mathrm{d}\varphi] + i\left(\frac{\Delta_2}{\Delta_1}\right)^{1/4} [\sin\psi\,\mathrm{d}\theta - \tan\theta\cos\psi\,\mathrm{d}\varphi] \right) \\ & \times \left(\left(\frac{\Delta_1}{\Delta_2}\right)^{1/4} [\cos\psi\,\mathrm{d}\theta + \tan\theta\sin\psi\,\mathrm{d}\varphi] - i\left(\frac{\Delta_2}{\Delta_1}\right)^{1/4} [\sin\psi\,\mathrm{d}\theta - \tan\theta\cos\psi\,\mathrm{d}\varphi] \right), \end{aligned} \tag{4.2}$$

where $\psi = \omega - \varphi$ as previously — the coordinates are $(r, \omega, \varphi, \theta)$.

Possible candidates for the generalized parafermions can be read-off from expression (4.2) up to phases and possibly terms that cancel out. We introduce the following candidates for *elliptic compact parafermions:*

$$\begin{aligned} \Psi^{(1)}_{\pm} = & \left[\left(\frac{\Delta_1}{\Delta_2}\right)^{1/4} (\cos(\omega-\varphi)\partial_+\theta + \tan\theta\sin(\omega-\varphi)\partial_+\varphi) \right. \\ & \left. \pm i \left(\frac{\Delta_2}{\Delta_1}\right)^{1/4} (\sin(\omega-\varphi)\partial_+\theta - \tan\theta\cos(\omega-\varphi)\partial_+\varphi) \right] \mathrm{e}^{\mp i(\omega+\phi_1)}\ , \\ \bar{\Psi}^{(1)}_{\pm} = & \left[\left(\frac{\Delta_1}{\Delta_2}\right)^{1/4} (\cos(\omega-\varphi)\partial_-\theta + \tan\theta\sin(\omega-\varphi)\partial_-\varphi) \right. \\ & \left. \mp i \left(\frac{\Delta_2}{\Delta_1}\right)^{1/4} (\sin(\omega-\varphi)\partial_-\theta - \tan\theta\cos(\omega-\varphi)\partial_-\varphi) \right] \mathrm{e}^{\pm i(\omega-\phi_1)}\ , \end{aligned} \tag{4.3}$$

as well as their non-compact companions:

$$\begin{aligned} \Psi^{(2)}_{\pm} &= \left[\frac{\partial_+ r}{r(\Delta_1\Delta_2)^{1/4}} \mp i(\Delta_1\Delta_2)^{1/4}\partial_+\omega \right] \mathrm{e}^{\mp i(\omega+\phi_2)}\ , \\ \bar{\Psi}^{(2)}_{\pm} &= \left[\frac{\partial_- r}{r(\Delta_1\Delta_2)^{1/4}} \pm i(\Delta_1\Delta_2)^{1/4}\partial_-\omega \right] \mathrm{e}^{\pm i(\omega-\phi_2)}\ . \end{aligned} \tag{4.4}$$

A noticeable difference with respect to the usual coset parafermions is that compact and non-compact ones are no longer decoupled in the ellipsis. This is expected since the metric (2.18) is not factorized into two parts.

The phases ϕ_1 and ϕ_2 can still be written as in (3.11). However, the currents $J^1_{\pm}$ and $J^2_{\pm}$ are no longer those in (3.12). They must coincide in the limit $\Delta_1 \to \Delta_2$ where we recover the circle, in order for the elliptic parafermions (4.3)–(4.4) to match the ordinary parafermions (3.10) and (3.16) in this limit:

$$\begin{aligned} \lim_{\Delta_1\to\Delta_2} J^1_{\pm} &= g_{\varphi\theta}|_{\Delta_1=\Delta_2}\partial_\pm\theta + g_{\varphi\varphi}|_{\Delta_1=\Delta_2}\partial_\pm\varphi = \tan^2\theta\,\partial_\pm\varphi\ , \\ \lim_{\Delta_1\to\Delta_2} J^2_{\pm} &= g_{\omega\omega}|_{\Delta_1=\Delta_2}\partial_\pm\omega = \coth^2\rho\,\partial_\pm\omega\ , \end{aligned} \tag{4.5}$$

that is we obtain the current components of (3.12). We will come back to the determination of $J^1_\pm$ and $J^2_\pm$ when analyzing the chirality properties, in section 4.2.

4.2 The deformation

In order to determine the would-be marginal operator at generic ℓ_i's, we must analyze how the action behaves under deformations of the ellipsis. Using the elliptic parafermions introduced previously, the lagrangian of the sigma model in the background (4.2) takes the form

$$\begin{aligned}\mathcal{L} &= (G_{\mu\nu}+B_{\mu\nu})\,\partial_+ x^\mu\,\partial_- x^\nu \\ &= \frac{1}{2}\left(\Psi^{(1)}_+\bar{\Psi}^{(1)}_+ \mathrm{e}^{2i\phi_1} + \Psi^{(1)}_-\bar{\Psi}^{(1)}_- \mathrm{e}^{-2i\phi_1} + \Psi^{(2)}_+\bar{\Psi}^{(2)}_+ \mathrm{e}^{2i\phi_2} + \Psi^{(2)}_-\bar{\Psi}^{(2)}_- \mathrm{e}^{-2i\phi_2}\right). \end{aligned} \tag{4.6}$$

Furthermore, the energy-momentum tensor takes again the form (3.17).

One can analyze the behavior of $\mathcal{L}$ under a general deformation of the ellipsis, $\ell_1 \to \ell_1+\delta\ell_1$ and $\ell_2 \to \ell_2+\delta\ell_2$. This amounts to considering $\Delta_1 \to \Delta_1+\delta\Delta_1$ and $\Delta_2 \to \Delta_2+\delta\Delta_2$. The variation of the action (4.6) is bilinear in the elliptic parafermions, as the action itself:

$$\begin{aligned}\delta\mathcal{L}|_{\delta\Delta_1,\delta\Delta_2} &= \frac{1}{4}\left(\frac{\delta\Delta_1}{\Delta_1}-\frac{\delta\Delta_2}{\Delta_2}\right)\left(\Psi^{(1)}_+\bar{\Psi}^{(1)}_- \mathrm{e}^{2i\omega} + \Psi^{(1)}_-\bar{\Psi}^{(1)}_+ \mathrm{e}^{-2i\omega}\right) \\ &\quad -\frac{1}{4}\left(\frac{\delta\Delta_1}{\Delta_1}+\frac{\delta\Delta_2}{\Delta_2}\right)\left(\Psi^{(2)}_+\bar{\Psi}^{(2)}_- \mathrm{e}^{2i\omega} + \Psi^{(2)}_-\bar{\Psi}^{(2)}_+ \mathrm{e}^{-2i\omega}\right). \end{aligned} \tag{4.7}$$

Similarly to the discussion in section 3.1, some of this general deformation turns out to be equivalent to a coordinate transformation $r \to r+\delta r$. One easily sees that for rescalings $\delta r = \varepsilon r$, the variation of the action $\delta\mathcal{L}|_{\text{repar}}$ can be written in a factorized form, similar to (4.7). By appropriately tuning ε versus $\delta\ell_1$ and $\delta\ell_2$, one can possibly simplify $\delta\mathcal{L}|_{\text{tot}} = \delta\mathcal{L}|_{\text{repar}} + \delta\mathcal{L}|_{\delta\Delta_1,\delta\Delta_2}$. In particular, $\delta\mathcal{L}|_{\text{tot}}$ may vanish for $\ell_2\delta\ell_1 = \ell_1\delta\ell_2$, which corresponds to deformations that do not alter the shape of the ellipsis (like $\delta\ell_1 = \delta\ell_2$ respects the circle $\ell_1 = \ell_2$).

We will not further explore the general structure of $\delta\mathcal{L}|_{\text{tot}}$. The reason is that the generalized parafermions lack two basic properties which seem incompatible. First, since for a generic ellipsis one breaks one of the U(1)'s available for the circle, only the combination $\zeta = \partial_\varphi + \partial_\omega$ remains a Killing vector with

$$J^\zeta_\pm = g_{\varphi\theta}\partial_\pm\theta + g_{\varphi\varphi}\partial_\pm\varphi + g_{\omega\omega}\partial_\pm\omega, \tag{4.8}$$

satisfying the standard conservation equation $\partial_+ J^\zeta_- + \partial_- J^\zeta_+ = 0$. Hence it is not clear which would be the independent phases ϕ_1 and ϕ_2 that enter in the definition of the generalized parafermions. The second problem is that the candidates for generalized elliptic parafermions cannot be made chiral and anti-chiral. That is neither $\partial_-\Psi^{(1\,\text{or}\,2)}_\pm$ nor $\partial_+\bar{\Psi}^{(1\,\text{or}\,2)}_\pm$ vanish.

Any further classical investigation is of little relevance. We have good reasons to believe that the operators which deform the ellipsis at a generic point are exactly marginal, not factorizable though. In order to prove this statement, we must compute their anomalous dimensions. This is left for future work.

JHEP01(2006)167

5. Conclusions

We would like to summarize our results. Our starting point was a geometric plus dilaton and antisymmetric background, generated by NS5-branes spread over an ellipsis. When the ellipsis degenerates onto a circle, we know that the background is described in terms of an exact CFT, T-dual to an arbifold of $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)\times\mathrm{SU}(2)/\mathrm{U}(1)$. We have exhibited (both in the direct and in the T-dual) the $(1,1)$ conformal operator responsible for triggering the deformation of the circle into an ellipsis. This operator is of a new type because it cannot be factorized in holomorphic times anti-holomorphic dimension-one currents. On the contrary, the marginal operator appears as a product of holomorphic and anti-holomorphic parafermions, dressed by a non-left-right-factorized function of the non-compact fields. We have proven in the purely bosonic case that the dressing allows for precisely adjusting the total conformal dimensions to $(1,1)$.

We have finally investigated the existence of a marginal generator all the way in the elliptic deformation. Our analysis is performed at a classical level and exhibits a natural generalization of the dimension-two operator of the circle. The latter is based on what we called *elliptic* parafermions that we have introduced and which mix the compact with the non-compact directions.

Although the computation of the anomalous dimension of the advertised marginal operator remains to be done, its generic structure strongly suggests that the background generated by NS5-branes distributed over an ellipsis is the target space of an exact conformal sigma model. The latter includes among other limiting cases the Eguchi-Hanson geometry. Therefore, besides the relevance of our approach in the framework of conformal field theory, it also opens up new possibilities for description of physically relevant string backgrounds.

Acknowledgments

We would like to thank V. Dotsenko, D. Orlando and D. Zoakos for scientific discussions.
P.M. P. acknowledges the University of Patras for kind hospitality as well partial financial support by the INTAS contract 03-51-6346 and by the EU under the contracts MEXT-CT-2003-509661, MRTN-CT-2004-005104 and MRTN-CT-2004-503369.
K. S. acknowledges CERN and the University of Neuchâtel for hospitality and financial support during part of this research. In addition partial support is provided through the European Community's program "Constituents, Fundamental Forces and Symmetries of the Universe" with contract MRTN-CT-2004-005104, the INTAS contract 03-51-6346 "Strings, branes and higher-spin gauge fields", the Greek Ministry of Education programs ΠΥΘΑΓΟΡΑΣ with contract 89194 and the program ΕΠΑΝ with code-number B.545.

References

[1] C.G. Callan Jr., J.A. Harvey and A. Strominger, *World sheet approach to heterotic instantons and solitons*, *Nucl. Phys.* **B 359** (1991) 611.

[2] A. Strominger, *Heterotic solitons*, *Nucl. Phys.* **B 343** (1990) 167.

[3] I. Antoniadis, S. Ferrara and C. Kounnas, *Exact supersymmetric string solutions in curved gravitational backgrounds*, *Nucl. Phys.* **B 421** (1994) 343 [hep-th/9402073].

[4] J.M. Maldacena and A. Strominger, *Semiclassical decay of near-extremal fivebranes*, *JHEP* **12** (1997) 008 [hep-th/9710014].

[5] K. Sfetsos, *Branes for Higgs phases and exact conformal field theories*, *JHEP* **01** (1999) 015 [hep-th/9811167].

[6] K. Sfetsos, *Rotating ns5-brane solution and its exact string theoretical description*, *Fortschr. Phys.* **48** (2000) 199 [hep-th/9903201].

[7] S. Förste, *A truly marginal deformation of* $\mathrm{SL}(2,r)$ *in a null direction*, *Phys. Lett.* **B 338** (1994) 36 [hep-th/9407198].

[8] S. Förste and D. Roggenkamp, *Current current deformations of conformal field theories and WZW models*, *JHEP* **05** (2003) 071 [hep-th/0304234].

[9] D. Israel, C. Kounnas and M.P. Petropoulos, *Superstrings on ns5 backgrounds, deformed* AdS_3 *and holography*, *JHEP* **10** (2003) 028 [hep-th/0306053].

[10] D. Israel, *Quantization of heterotic strings in a goedel/anti de Sitter spacetime and chronology protection*, *JHEP* **01** (2004) 042 [hep-th/0310158].

[11] D. Israel, C. Kounnas, D. Orlando and P.M. Petropoulos, *Electric / magnetic deformations of* S^3 *and* AdS_3 *and geometric cosets*, *Fortschr. Phys.* **53** (2005) 73 [hep-th/0405213].

[12] P.M. Petropoulos, *Deformations and geometric cosets*, hep-th/0412328.

[13] D. Israel, C. Kounnas, D. Orlando and P.M. Petropoulos, *Heterotic strings on homogeneous spaces*, *Fortschr. Phys.* **53** (2005) 1030 [hep-th/0412220].

[14] E. Kiritsis, C. Kounnas, P.M. Petropoulos and J. Rizos, *Five-brane configurations without a strong coupling regime*, *Nucl. Phys.* **B 652** (2003) 165 [hep-th/0204201].

[15] A. Giveon and D. Kutasov, *Little string theory in a double scaling limit*, *JHEP* **10** (1999) 034 [hep-th/9909110].

[16] E. Kiritsis, C. Kounnas, P.M. Petropoulos and J. Rizos, *Five-brane configurations, conformal field theories and the strong-coupling problem*, hep-th/0312300.

[17] D. Israel, C. Kounnas, A. Pakman and J. Troost, *The partition function of the supersymmetric two-dimensional black hole and little string theory*, *JHEP* **06** (2004) 033 [hep-th/0403237].

[18] T. Eguchi and A.J. Hanson, *Selfdual solutions to euclidean gravity*, *Ann. Phys. (NY)* **120** (1979) 82.

[19] I. Bakas and K. Sfetsos, *States and curves of five-dimensional gauged supergravity*, *Nucl. Phys.* **B 573** (2000) 768 [hep-th/9909041].

[20] I. Bakas, A. Brandhuber and K. Sfetsos, *Domain walls of gauged supergravity, m-branes and algebraic curves*, *Adv. Theor. Math. Phys.* **3** (1999) 1657 [hep-th/9912132].

[21] T.H. Buscher, *A symmetry of the string background field equations*, *Phys. Lett.* **B 194** (1987) 59.

[22] I. Bakas and K. Sfetsos, *Gravitational domain walls and p-brane distributions*, *Fortschr. Phys.* **49** (2001) 419 [hep-th/0012125].

[23] M.K. Prasad, *Equivalence of Eguchi-hanson metric to two-center Gibbons-Hawking metric*, *Phys. Lett.* **B 83** (1979) 310.

[24] K. Sfetsos, *Duality and restoration of manifest supersymmetry*, *Nucl. Phys.* **B 463** (1996) 33 [hep-th/9510034].

[25] G.W. Gibbons and S.W. Hawking, *Gravitational multi - instantons*, *Phys. Lett.* **B 78** (1978) 430.

[26] V.A. Fateev and A.B. Zamolodchikov, *Parafermionic currents in the two-dimensional conformal quantum field theory and selfdual critical points in $Z(n)$ invariant statistical systems*, *Sov. Phys. JETP* **62** (1985) 215.

[27] J.D. Lykken, *Finitely reducible realizations of the $N = 2$ superconformal algebra*, *Nucl. Phys.* **B 313** (1989) 473.

[28] K. Bardacki, M.J. Crescimanno and E. Rabinovici, *Parafermions from coset models*, *Nucl. Phys.* **B 344** (1990) 344.

[29] S. Chaudhuri and J.D. Lykken, *String theory, black holes and* $\mathrm{SL}(2,\mathbb{R})$ *current algebra*, *Nucl. Phys.* **B 396** (1993) 270 [hep-th/9206107].

[30] C. Kounnas, *Four-dimensional gravitational backgrounds based on $N = 4$, $c = 4$ superconformal systems*, *Phys. Lett.* **B 321** (1994) 26 [hep-th/9304102].

[31] V.A. Fateev and A.B. Zamolodchikov, *Integrable perturbations of $Z(n)$ parafermion models and $O(3)$ sigma model*, *Phys. Lett.* **B 271** (1991) 91.

[32] L. Alvarez-Gaumé and D.Z. Freedman, *Geometrical structure and ultraviolet finiteness in the supersymmetric sigma model*, *Commun. Math. Phys.* **80** (1981) 443.

[33] J. Gates, S. J., C.M. Hull and M. Roček, *Twisted multiplets and new supersymmetric nonlinear sigma models*, *Nucl. Phys.* **B 248** (1984) 157.

[34] B. de Wit and P. van Nieuwenhuizen, *Rigidly and locally supersymmetric two-dimensional nonlinear sigma models with torsion*, *Nucl. Phys.* **B 312** (1989) 58.

[35] P.S. Howe and G. Papadopoulos, *Further remarks on the geometry of two-dimensional nonlinear sigma models*, *Class. and Quant. Grav.* **5** (1988) 1647.

[36] D. Tong, *Ns5-branes, t-duality and worldsheet instantons*, *JHEP* **07** (2002) 013 [hep-th/0204186].

[37] S. Chaudhuri and J.A. Schwartz, *A criterion for integrably marginal operators*, *Phys. Lett.* **B 219** (1989) 291.

[38] S.F. Hassan and A. Sen, *Marginal deformations of WZNW and coset models from $O(d,d)$ transformation*, *Nucl. Phys.* **B 405** (1993) 143 [hep-th/9210121].

JHEP01(2006)167